

Remark on an elastic plate interacting with a gas in a semi-infinite tube: periodic solutions

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Abstract

We consider a conservative system consisting of an elastic plate interacting with a gas filling a semi-infinite tube. The plate is placed on the bottom of the tube. The dynamics of the gas velocity potential is governed by the linear wave equation. The plate displacement satisfies the linear Kirchhoff equation. We show that this system possesses an infinite number of periodic solutions with the frequencies tending to infinity. This means that the well-known property of decaying of local wave energy in tube domains does not hold for the system considered.

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Recently there was great interest in the study of long-time dynamics of elastic plates interacting with a flow of gas (see, e.g., [3, 4, 6, 8, 9, 10, 11, 13, 14] and the literature cited in those sources). The corresponding model has the form

$$\begin{cases} u_{tt} + \Delta^2 u + \gamma u_t + f(u) = (\partial_t + U \partial_x) [\phi|_{x_3=0}] & \text{in } \Omega \times \mathbb{R}_+, \\ u(0) = u_0; \quad u_t(0) = u_1, \\ u = \Delta u = 0 & \text{on } \partial\Omega \times \mathbb{R}_+, \end{cases} \quad (1)$$

where $\phi(x_1, x_2, x_3; t)$ solves the problem in $\mathbb{R}_+^3 \equiv \{x = (x_1; x_2; x_3) : x_3 > 0\}$:

$$\begin{cases} (\partial_t + U \partial_x)^2 \phi = \Delta \phi & \text{in } \mathbb{R}_+^3 \times \mathbb{R}_+, \\ \phi(0) = \phi_0; \quad \phi_t(0) = \phi_1 & \text{in } \mathbb{R}_+^3 \\ \partial_{x_3} \phi = \begin{cases} (\partial_t + U \partial_x) u(x_1, x_2) & \text{on } \Omega \times \mathbb{R}_+, \\ 0 & \text{on } (\mathbb{R}^2 \setminus \overline{\Omega}) \times \mathbb{R}_+. \end{cases} \end{cases} \quad (2)$$

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Here above Ω is a bounded smooth domain in \mathbb{R}^2 identified with

$$\{(x_1; x_2; 0) : (x_1; x_2) \in \Omega\} \subset \overline{\mathbb{R}_+^3}.$$

The term $f(u)$ describes a nonlinear force which can be (a) von Karman type (like as in [6]), or (b) Berger type (like as in [1]), or even (c) generated by some Nemytskii operator (see, e.g., [5, 7]). The unknown function $u = u(x_1, x_2; t)$ measures the transverse displacement of the plate at the point $(x_1; x_2)$ and time t . The boundary conditions for u means that the plate is hinged on its edge. The function $\phi(x, t) = \phi(x_1, x_2, x_3; t)$ is velocity potential of the gas filling the domain \mathbb{R}_+^3 . Here we deal with interaction of a plate with a gas flow moving with the speed U in the direction of the axis x_1 . The aerodynamical pressure of the gas on the plate is given by the term $p(x, t) = \nu(\phi_t + U\phi_{x_1})|_{x_3=0}$, the parameter $\nu > 0$ characterizes the intensity of the interaction between the gas and the plate. The transverse displacement $u(x, t)$ has influence on the gas via boundary condition in (2).

For recent surveys of mathematical and applied aspects of the model above we refer to [3, 4]. Here we only mention the convergence results in the subsonic case ($0 \leq U < 1$) which were established in [13, 14] (see also [6] for related facts) and state stabilization of solutions to stationary states of the system when $t \rightarrow \infty$ under some conditions concerning initial data of the gas velocity potential. The corresponding argument requires positivity of the damping parameter γ and involves a gradient-type structure of the system in the case considered.

In this relation the question (see [2] and [6, p.694]) arises whether it is possible to obtain a similar stabilization result in the absence ($\gamma = 0$) of the internal damping in the plate. This conjecture is based on the well-known property of the local energy decay for the wave equation in \mathbb{R}^3 and some other unbounded domains (see also Proposition 2.1 below).

Our main goal in this note is to show that unboundness of the wave domain \mathcal{O} is not sufficient to guarantee stabilization of solutions to equilibria. For this we consider a linear plate model without any damping ($\gamma = 0$), coupled to the flow via matching velocities. The parameter U is taken to be zero. We show that in this scenario, periodic solutions may exist. In the case of a specific tubular domain they are explicitly constructed. Whether the same result holds for nonlinear plate is an open question. However, the model indicates the necessity of introducing mechanical damping in the plate model *if one expects a strong convergence to equilibria* of the full flow-structure system.

1 Model

Let Ω be a smooth bounded domain in \mathbb{R}^2 . We consider the following problem

$$\begin{cases} \partial_t^2 u + \Delta^2 u - \nu \cdot \partial_t [\phi|_{x_3=0}] = 0, & x = (x_1; x_2) \in \Omega, \ t > 0, \\ u|_{\partial\Omega} = \Delta u|_{\partial\Omega} = 0, & t > 0, \\ u|_{t=0} = u_0(x), \quad \partial_t u|_{t=0} = u_1(x), & x = (x_1; x_2) \in \Omega, \end{cases} \quad (3)$$

We denote by $\phi|_{x_3=0}$ the trace of a function $\phi(x_1, x_2, x_3, t)$ in $\Omega \times \mathbb{R}_+ \times \mathbb{R}_+$ which solves the problem

$$\begin{cases} \partial_{tt}\phi = \Delta\phi, & x \in \mathcal{O}_+ \equiv \{x = (x_1; x_2; x_3) : (x_1; x_2) \in \Omega, x_3 > 0\}, \\ \partial_{x_3}\phi = \partial_t u(x_1, x_2, t), & (x_1; x_2) \in \Omega, x_3 = 0, t > 0, \\ \phi = 0, & (x_1; x_2) \in \partial\Omega, x_3 > 0, t > 0, \\ \phi|_{t=0} = \phi_0(x), & \partial_t\phi|_{t=0} = \phi_1(x), \quad x \in \mathcal{O}_+. \end{cases} \quad (4)$$

The function $u = u(x, t)$ is the transverse displacement of the hinged plate. The function $\phi(x, t) = \phi(x_1, x_2, x_3; t)$ is velocity potential of the gas filling the tube \mathcal{O}_+ . The pressure of the gas on the plate is given by the term $p(x, t) = \nu \cdot \phi_t|_{x_3=0}$, where $\nu > 0$.

In the case of bounded domains \mathcal{O} systems like (3) and (4) was studied by many authors (see the discussion and the references in [6] and [12]).

Below we use the notation $H^s(D)$ for the Sobolev space of order s on a domain D in \mathbb{R}^d , $d = 2, 3$.

We start with the following assertion.

Proposition 1.1 (Well-posedness). *Assume that*

$$\phi_0 \in H^1(\mathcal{O}_+), \quad \phi_1 \in L_2(\mathcal{O}_+), \quad u_0 \in (H^2 \cap H_0^1)(\Omega), \quad u_1 \in L_2(\Omega).$$

Then there exists a unique couple $\{u; \phi\}$ of function

$$u \in C^1(\mathbb{R}_+; L_2(\Omega)) \cap C(\mathbb{R}_+; (H^2 \cap H_0^1)(\Omega))$$

and

$$\phi \in C^1(\mathbb{R}_+; L_2(\mathcal{O}_+)) \cap C(\mathbb{R}_+; H^1(\mathcal{O}_+))$$

solving (3) and (4) in the sense of distributions. Moreover this solution satisfies the energy preservation law of the form

$$\mathcal{E}(t) \equiv E_{\mathcal{O}_+}^{gas}(\phi_t(t), \phi(t)) + \nu^{-1} E_{\Omega}^{plate}(u_t(t), u(t)) = \mathcal{E}(0), \quad \text{for all } t > 0,$$

where we use the notations

$$E_D^{gas}(\phi_t, \phi) = \frac{1}{2} \int_D [|\phi_t(x, t)|^2 + |\nabla \phi(x, t)|^2] dx \quad (5)$$

for every $D \subseteq \mathcal{O}_+ \subset \mathbb{R}^3$ and

$$E_{\Omega}^{plate}(u_t, u) = \frac{1}{2} \int_{\Omega} [|u_t(x, t)|^2 + |\Delta u(x, t)|^2] dx.$$

Proof. We can apply well-known general result reported in [12], see also [6, Chapter 6], where nonlinear versions of similar problems are discussed. However we can also give more direct argument based on some symmetry of this linear

problem and involving the variables separation. We sketch the corresponding argument below.

Let $\{e_k\}$ be an orthonormal basis in $L_2(\Omega)$ consisting of eigenvectors of the problem

$$\Delta w + \lambda w = 0, \quad w|_{\partial\Omega} = 0, \quad (6)$$

and $0 < \lambda_1 \leq \lambda_2 \leq \dots$ the corresponding eigenvalues. We are looking for solutions to problem (3) and (4) in the following form

$$u(x_1, x_2, t) = \sum_{k=1}^{\infty} u_k(t) e_k(x_1, x_2), \quad \phi(x_1, x_2, x_3, t) = \sum_{k=1}^{\infty} \phi_k(t, x_3) e_k(x_1, x_2). \quad (7)$$

It is clear that the function $\phi_k(t, z)$ solves the problem

$$\begin{cases} \partial_{tt}\phi_k - \partial_{zz}\phi_k + \lambda_k\phi_k = 0, & z > 0, t > 0, \\ \partial_z\phi_k = \dot{u}_k(t), & z = 0, t > 0, \\ \phi_k|_{t=0} = \phi_{0k}(z), \quad \partial_t\phi_k|_{t=0} = \phi_{1k}(z), & z > 0, \end{cases} \quad (8)$$

where $u_k(t)$ satisfies the equation

$$\ddot{u}_k + \lambda_k^2 u_k - \nu \partial_t \phi_k(0, t) = 0, \quad u_k|_{t=0} = u_{0k}, \quad \dot{u}_k|_{t=0} = u_{1k}. \quad (9)$$

It is easy to show that for each k problem (8) and (9) has a unique solution $(\phi_k(t), u_k(t))$ for which we have the following energy balance relation

$$E_k(t) = E_k(s), \quad t \geq s,$$

where the energy E_k of the k -mode has the form

$$\begin{aligned} E_k(t) = & \frac{1}{2} \int_0^\infty [|\partial_t \phi_k(t, z)|^2 + |\partial_z \phi_k(t, z)|^2 + \lambda_k |\phi_k(t, z)|^2] dz \\ & + \frac{1}{2\nu} [|\dot{u}_k|^2 + \lambda_k^2 |u_k|^2]. \end{aligned}$$

This observations allow us to obtain appropriate a priori estimates and conclude the proof by the standard compactness method. \square

2 Dynamics

We start with the following assertion that shows a local energy decay in the case when the bottom Ω of the cylinder \mathcal{O}_+ is *rigid*. This means that we consider

the wave dynamics only. This dynamics is described by the following equations

$$\left\{ \begin{array}{l} \partial_{tt}\phi = \Delta\phi, \quad x \in \mathcal{O}_+, \\ \partial_{x_3}\phi = 0, \quad (x_1; x_2) \in \Omega, \quad x_3 = 0, \quad t > 0, \\ \phi = 0, \quad (x_1; x_2) \in \partial\Omega, \quad x_3 > 0, \quad t > 0, \\ \phi|_{t=0} = \phi_0(x), \quad \partial_t\phi|_{t=0} = \phi_1(x), \quad x \in \mathcal{O}_+. \end{array} \right. \quad (10)$$

Proposition 2.1 (Local energy decay). *Assume that $\phi_0 \in H^1(\mathcal{O}_+)$ and $\phi_1 \in L_2(\mathcal{O}_+)$. Then problem (10) has a unique variational solution ϕ which belongs to the class*

$$\phi \in C^1(\mathbb{R}_+; L_2(\mathcal{O}_+)) \cap C(\mathbb{R}_+; H^1(\mathcal{O}_+)).$$

This this solution satisfies the energy preservation law of the form

$$E_{\mathcal{O}_+}^{gas}(\phi_t(t), \phi(t)) = E_{\mathcal{O}_+}^{gas}(\phi_1, \phi_0), \quad \text{for all } t > 0.$$

Moreover, we have decaying of ϕ as $t \rightarrow \infty$ in the local energy norm, i.e.,

$$\lim_{t \rightarrow +\infty} E_{\mathcal{O}_+^R}^{gas}(\phi_t(t), \phi(t)) = 0 \quad \text{for every } R > 0, \quad (11)$$

where $E_{\mathcal{O}_+^R}^{gas}$ is given by (5) with

$$\mathcal{O}_+^R \equiv \{x = (x_1; x_2; x_3) : (x_1; x_2) \in \Omega, \quad 0 < x_3 < R\}.$$

Proof. The existence and uniqueness of solutions to (10) is obvious (we can use the same idea as in Proposition 1.1, for instance). It is also clear that the energy relation is satisfied. Thus we only need to establish the property in (11).

Extending the initial data as even functions in the variable x_3 on the whole x_3 -axis we can consider the wave equation in the domain

$$\mathcal{O} = \{x = (x_1; x_2; x_3) : (x_1; x_2) \in \Omega, \quad -\infty < x_3 < +\infty\}.$$

with the Dirichlet boundary conditions on $\partial\mathcal{O}$. Now we can separate variables as above and apply the same idea as in [15] to prove (11) for *localized* initial data ϕ_0 and ϕ_1 . In fact the article [15] contains exactly this statement for the case when $\Omega = (0, \pi) \times (0, \pi)$. The method suggested in [15] relies on the presentation of the solution ϕ in the form

$$\phi(x_1, x_2, x_3, t) = \sum_{m,n=1}^{\infty} \phi_{mn}(t, x_3) e_{mn}(x_1, x_2),$$

where $e_{mn}(x_1, x_2) = 2\pi^{-1} \sin ma_1 \sin nx_2$ are solutions to the spectral problem (6) for $\Omega = (0, \pi) \times (0, \pi)$ and $\phi_{mn}(t, z)$ solves the equation

$$\partial_{tt}\phi_{mn} - \partial_{zz}\phi_{mn} + (m^2 + n^2)\phi_{mn} = 0, \quad z \in \mathbb{R}, \quad t > 0,$$

Establishing appropriate bounds for ϕ_{mn} (see [15]) one can prove the desired result for $\Omega = (0, \pi) \times (0, \pi)$. The calculations given in [15] can be easily extended to the case of general domains Ω .

Then using approximation procedure for initial data and the energy relation we can obtain the result for every pair $(\phi_1; \phi_0) \in L_2(\mathcal{O}_+) \times H^1(\mathcal{O}_+)$. \square

The decay property of the local wave energy demonstrated in Proposition 2.1 is not valid for the coupled system in (3) and (4). More precisely, we show that problem (3) and (4) possesses infinite number of periodic solutions with different periods.

Theorem 2.2. *Let $\{\lambda_k\}$ be the eigenvalues of problem (6) and $\{e_k\}$ be the corresponding eigen-basis. Then there exist sequences $\{\omega_k\}$ and $\{\alpha_k\}$ of positive numbers with properties*

$$\lim_{k \rightarrow \infty} [\omega_k^2 - \lambda_k] = 0 \quad \text{and} \quad \lim_{k \rightarrow \infty} \alpha_k \lambda_k = \nu \quad (12)$$

such that the functions

$$\varphi^k(x_1, x_2, x_3; t) = [A_k \cos \omega_k t + B_k \sin \omega_k t] e^{-\alpha_k x_3} e_k(x_1, x_2) \quad (13)$$

and

$$u^k(x_1, x_2; t) = -\frac{\alpha_k}{\omega_k} [A_k \sin \omega_k t - B_k \cos \omega_k t] e_k(x_1, x_2), \quad (14)$$

where A_k and B_k are arbitrary real numbers, solve problem (3) and (4) with appropriate initial data. Each trajectory $(\varphi^k; \varphi_t^k; u^k; u_t^k)$ is Lyapunov stable in the phase space $\mathcal{H} = H^1(\mathcal{O}_+) \times L_2(\mathcal{O}_+) \times H^1(\Omega) \times L_2(\Omega)$.

Proof. Let us look for solutions to (8) and (9) of the form

$$\phi_k(t, z) = e^{i\omega t} e^{-\alpha z}, \quad u_k(t) = a e^{i\omega t}$$

with $\alpha > 0$ and $\omega, a \in \mathbb{C}$. The substitution in (8) and (9) gives us the relations

$$\begin{aligned} -\omega^2 - \alpha^2 + \lambda_k &= 0, \\ \alpha &= i\omega a, \\ a(-\omega^2 + \lambda_k^2) + i\nu\omega &= 0. \end{aligned}$$

This implies that $a = -i\alpha\omega^{-1}$ and also

$$\omega^2 = \lambda_k - \alpha^2, \quad \omega^2 = \frac{\alpha}{\alpha + \nu} \lambda_k^2. \quad (15)$$

One can see for every k there exists unique solution (ω_k^2, α_k) to (15). It is also easy to find that

$$(\omega_k^2, \alpha_k) \sim (\lambda_k, \nu\lambda_k^{-1}) \quad \text{when } k \rightarrow +\infty$$

in the sense of (12). This implies the structure of a solution written in (13) and (14).

Stability properties of solutions follow from the energy preservation law. \square

Theorem 2.2 shows that the elasticity of the bottom Ω of the cylinder \mathcal{O}_+ destroy the local energy decay property which we observe in the case of rigid bottom (see Proposition 2.1).

We conclude this note with several open questions which, we believe, are important for understanding of long-time dynamics of flow-structure systems.

Open Questions:

- Can we show that the minimal subspace in \mathcal{H} containing all solutions $(\varphi^k; \varphi_t^k; u^k; u_t^k)$ is asymptotically stable? Is this subspace a global minimal attractor? If not, what is a real candidate on the role of global attractor for (3) and (4)?
- What can we say about stability and spectral properties of the generators of C_0 semigroups generated by (3) and (4) and its dissipative perturbation? For instance, is it possible to stabilize the system by introducing internal damping in the plate component only?

These questions are important not only for linear dynamics, but also for non-linear perturbations of (3) and (4).

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